

On the topology of the inverse limit of a branched covering over a Riemann surface.

Carlos Cabrera, Chokri Cherif and Avraham Goldstein

March 1, 2012

Abstract

We introduce the Plaque Topology on the inverse limit of a branched covering map over a Riemann surface to study its dynamical properties. We also consider a Boolean Algebra to compute local topological invariants. With these tools we obtain a description of the points in the inverse limit.

1 Introduction

An inverse dynamical system is a sequence:

$$S_1 \xleftarrow{f_1} S_2 \xleftarrow{f_2} S_3 \dots$$

of Riemann Surfaces S_i and branched coverings $f_i : S_{i+1} \rightarrow S_i$ where all S_i are equal to a given Riemann surface S_0 and all f_i are equal to a given branched covering map $f : S_0 \rightarrow S_0$ of degree d . In this work, we assume that $1 < d < \infty$ and that S_0 is either the Unit Disc, the Complex Plane or the Riemann Sphere. We define the Plaque Inverse Limit [P.I.L.] S_∞ of f to be the set of all the sequences of points $x_1 \in S_1, x_2 \in S_2, \dots$, such that $f_i(x_{i+1}) = x_i$, equipped with the topology in which the open sets are all sequences of open sets $U = \{U_i \subset S_i \mid f_i(U_{i+1}) = U_i\}_{i=1, \dots}$. For $i = 1, 2, \dots$, let p_i be the map from S_∞ onto S_i , that takes $(x_1, x_2, \dots) \in S_\infty$ to $x_i \in S_i$. The maps p_i are continuous and satisfy $f_i \circ p_{i+1} = p_i$. These p_i are called the projection maps.

The standard Inverse Limit \tilde{S}_∞ of f , as a set, is defined exactly like P.I.L., but it is equipped with the Tychonoff topology in which the open sets are generated by the sets of the form $p_i^{-1}(U_i)$, where U_i is an open subset of S_i . The topology of P.I.L. has more open sets than the standard Inverse Limit and thus, the identity map from the P.I.L. onto the Inverse Limit is continuous.

Obviously, the projection maps p_i are continuous as maps from \tilde{S}_∞ onto S_i . Actually, the categorical definition of the topology of the inverse limit of an inverse system is precisely the “minimal” topology that makes the projection maps p_i continuous. Minimality in this context means that any other object, which comes with maps into the inverse system, commuting with the maps of the inverse system, can be mapped into the inverse system through the inverse limit. It is important to notice that the map f induces an automorphism \tilde{f} of the P.I.L. and an automorphism \tilde{f} of the inverse limit.

The local base of the topology of S_∞ at a point x consists of all open sets U containing x such that each U_i which is conformally equivalent to the unit disk in \mathcal{C} and f_i , restricted to U_{i+1} , is conformally equivalent to some polynomial map of a degree between 1 and d . Such open sets U are called plaques. When we speak of a neighborhood of a point in S_∞ we assume it to be a plaque. A point $x \in S_\infty$ is called regular if for some neighborhood U of x exists n such that U_{n+i+1} contains no critical points of f_{n+i} for all $i = 0, 1, 2, \dots$. Thus, $f_{n+i} : U_{n+i+1} \rightarrow U_{n+i}$ is a conformal equivalence in this instance. Otherwise, the point $x \in S_\infty$ is called irregular.

The set Δ of all regular points of S_∞ is open and each of its path-connected components has a Riemann Surface structure. These Riemann Surfaces, regarded as path-connected components of S_∞ , were studied and fully classified in [6]. In this work we:

- Show that Δ is not empty;
- Show that a point $x \in S_\infty$ is irregular if, and only if, there exists a neighborhood U of x such that, for any neighborhood V of x with its closure $\overline{V} \subset U$, the open set $V - \{x\}$ is an uncountable union of disjoint path-connected components. Thus, at an irregular point $x \in S_\infty$ the P.I.L. is not even a topological manifold;
- Develop and construct a σ -Algebraic machinery allowing us to obtain and compute certain local topological invariants of P.I.L.;
- Relate these local topological invariants of P.I.L. to the dynamical properties of the holomorphic Dynamical System S .

2 Constructions and Definitions

For a dynamical system $f : S_0 \rightarrow S_0$, where S_0 is either the Unit Disc, the Complex Plane or the Riemann Sphere and f is a branched covering of degree $1 < d < \infty$, we define an inverse dynamical system S as:

$$S_0 \xleftarrow{f} S_1 \xleftarrow{f_1} S_2 \xleftarrow{f_2} \dots$$

where $S_0 = S_1 = S_2 = \dots$ and $f = f_1 = f_2 = \dots$. We speak of f as a map from S_i to S_{i-1} for $i > 0$.

We denote the critical points of f by c_1, \dots, c_k and the critical set $\{c_1, \dots, c_k\}$ we denote by C .

Since $1 \leq k \leq \chi(S_0) \cdot (d - 1)$, where $\chi(S_0)$ is the Euler characteristic of S_0 , we get $1 \leq k \leq 2(d - 1)$ for the Riemann Sphere and $1 \leq k \leq (d - 1)$ for the Complex Plain and the Unit Disk.

Definition 1. *The Plaque Inverse Limit [P.I.L.] S_∞ of f is the set of all sequences of points $x_1 \in S_1, x_2 \in S_2, \dots$, such that $f_i(x_{i+1}) = x_i$, is equipped with the topology generated by sequences of open sets $U = U_1 \subset S_1, U_2 \subset S_2, \dots$ such that $f_i(U_{i+1}) = U_i$.*

The P.I.L. S_∞ is equipped with continuous projection maps $p_i : S_\infty \rightarrow S_i$, defined by $p_i(x_1, x_2, \dots) = x_i$. We have $f_i \circ p_{i+1} = p_i$.

Definition 2. An open set $U \subset S_\infty$ is called a plaque if each $U_i \subset S_i$ is conformally equivalent to the unit disk in \mathbb{C} and f_i , restricted to U_{i+1} , are conformally equivalent to a polynomial map of a degree $\leq d$.

All the plaques, containing a point $x \in S_\infty$, constitute a local base of the topology of S_∞ at x . In this work, all open neighborhoods to be considered, are assumed to be plaques.

Definition 3. A point $x \in S_\infty$ is called regular if there exists an open neighborhood U of x such that, for some positive integer n , all $f_{n+i} : U_{n+i+1} \rightarrow U_{n+i}$, in which $i = 0, 1, 2, \dots$, are bijections. Otherwise, x is called irregular.

The set of all regular points in S_∞ is denoted by Δ .

Lemma 4. The set Δ is not empty.

Proof. Let $q = 2d$ for the Riemann Sphere, and $q = d$ for the Complex Plane and the Unit Circle. Let us take any q nonempty, disjoint, simply connected open sets $U_1(1), \dots, U_1(q)$ in S_1 , which do not contain any images of points of the critical set C under f_1 . Then, for each $U_1(i)$, there are d disjoint, simply connected, open sets in S_2 which map onto $U_1(i)$ by f_1 . Amongst these, $d \cdot q$ open pre-image sets in S_2 , at least $2d^2 - 2(d-1)$ do not contain any images of points of the critical set C under f_2 for the Riemann Sphere and at least $d^2 - (d-1)$ do not contain any images of points of C under f_2 for either the Complex Plane or the Unit Disk. Since from $2 \leq d$ we get that $2d^2 - 2(d-1) \geq 2d$ and $d^2 - (d-1) \geq d$ we get at least q open, simply connected, pre-images $U_2(1), \dots, U_2(q)$ of [some of] $U_1(1), \dots, U_1(q)$ which do not contain any images of points of C under f_2 . Thus, we can repeat this process an infinite number of times. This produces a non-empty, open, simply connected set in S_∞ which contains no critical points of any of f_i . \square

Now we will introduce certain Algebraic structures, which are used to compute local topological invariants of the Plaque Inverse Limits. The set of all binary sequences with the below four points is a Boolean Algebra.

- the operations \vee and \wedge defined by performing the binary **or** and **and** operations, respectively, in each coordinate of the sequences;
- a partial order \leq on the sequences defined by: $b \leq a$ if $a \vee b = a$;
- with the order above, there is a minimal element $(0, 0, 0, \dots)$ and a maximal element $(1, 1, 1, \dots)$;
- the negation operation \neg , which interchanges 0 and 1 in every entry of the binary sequence;

Two binary sequences are considered almost equal if they differ only in a finite amount of places. Being almost equal, is an equivalence relation. It respects the \vee , \wedge , \neg operations, the partial order \leq and the minimal and maximal elements.

Definition 5. The set I is the Boolean Algebra of all classes of almost equal binary sequences, equipped with the \vee and \wedge operations which are defined as follows:

$$[a_1, a_2, \dots] \vee [b_1, b_2, \dots] = [a_1 \vee b_1, a_2 \vee b_2, \dots]$$

and

$$[a_1, a_2, \dots] \wedge [b_1, b_2, \dots] = [a_1 \wedge b_1, a_2 \wedge b_2, \dots].$$

Its minimal element is $\mathbf{0} = [0, 0, \dots]$ and its maximal element is $\mathbf{1} = [1, 1, \dots]$. Its negation is $\neg[a_1, a_2, \dots] = [\neg a_1, \neg a_2, \dots]$.

Definition 6. For every $a \in I$, we define $\alpha(a) \subset I$ as the set of all $b \in I$ such that $b \leq a$.

Note that $\alpha(a) \cup \alpha(b) \subset \alpha(a \vee b)$ and $\alpha(a \wedge b) = \alpha(a) \cap \alpha(b)$ and $\alpha(\mathbf{0}) = \{\mathbf{0}\}$ and $\alpha(\mathbf{1}) = I$. We write $\neg\alpha(a)$ as $\alpha(\neg a)$.

Definition 7. The sigma-lattice A , spanned by all $\alpha(a)$ in which $a \in I$, with the operations \cup and \cap and with the minimal element $\{\mathbf{0}\}$ and the maximal element I , is called the signature sigma-lattice. The elements of A are called signatures.

It is clear that \subset defines a partial order on A . This partial order is consistent with the partial order \leq of I under the map α , since if $b \leq a$ then $b \in \alpha(a)$ and therefore, $\alpha(b) \subset \alpha(a)$. Vice versa, if $\alpha(b) \subset \alpha(a)$ then $b \in \alpha(a)$ and therefore, $b \leq a$.

Definition 8. The map $\text{shift}_m : I \rightarrow I$, for any integer m , maps the class $[i] \in I$ to the class of the binary sequence i with m initial 0 entries adjoined to it if $m \geq 0$ and with m initial entries deleted from it if $m \leq 0$.

We have that $\text{shift}_0 = \text{Id}_I$ and $\text{shift}_m \circ \text{shift}_{-m} = \text{Id}_I$ for all m , since changing finite amount of entries in a binary sequence does not change its class in I .

Lemma 9. The maps $\text{shift}_m : I \rightarrow I$ induce maps $\text{shift}_m : A \rightarrow A$ and again $\text{shift}_0 = \text{Id}_A$ and $\text{shift}_m \circ \text{shift}_{-m} = \text{Id}_A$.

3 Properties of Lattices I and A .

In this work we need the following crucial property and its corollaries: Let $[i_1], [i_2], [i_3], \dots$ and $[t_1], [t_2], [t_3], \dots$ be elements of I .

Theorem 10. If

$$\alpha[i_1] \cup \alpha[i_2] \cup \alpha[i_3] \cup \dots = \beta = \alpha[t_1] \cap \alpha[t_2] \cap \alpha[t_3] \cup \dots$$

then there exist some natural numbers m and n such that

$$\alpha[i_m] = \beta = \alpha[t_1] \cap \dots \cap \alpha[t_n]$$

and so $[i_m] = [t_1] \wedge \dots \wedge [t_n]$.

Proof. We have that $[i_p] \leq [t_q]$ for any natural numbers p and q . Thus $\alpha([i_1] \vee \dots \vee [i_n]) \leq \alpha[t_1] \cap \dots \cap \alpha[t_n] = \alpha([t_1] \wedge \dots \wedge [t_n])$ for any finite n . We define $[i'_n] = [i_1] \vee \dots \vee [i_n]$ and $[t'_n] = [t_1] \wedge \dots \wedge [t_n]$ for all n . Then, $[i'_n] \leq [t'_n]$ for all n . For each $n = 1, 2, \dots$ we can, inductively on n , select some representatives i'_n and t'_n of the classes $[i'_n]$ and $[t'_n]$ such that $i'_n \leq t'_n$ and that $i'_{n-1} \leq i'_n$ and $t'_n \leq t'_{n-1}$ for all $n = 2, 3, \dots$. Let $z_n = t'_n - i'_n$ be the binary sequence which has 1 in all the places where t'_n has 1 and i'_n has 0 and 0 in all other places. Note that the classes $[z_1], [z_2], \dots$ do not depend on the choices of the representatives, which we made for the classes $[i'_n]$ and $[t'_n]$, so the subtraction operation $[t'_n] - [i'_n]$ is actually well defined in I . We have the inequalities $z_1 \geq z_2 \geq \dots$. If all z_n are not almost equal to $(0, 0, \dots)$, then let z be the binary string which has its first 1 in the same place where z_1 has its first 1 and its second 1 in the same place where z_2 has its second 1 and so on. All the other places of z contain 0. Thus z will have infinitely many 1 entries and for every n almost all these 1s, except some finite amount of them, will be in the places where t'_n has 1 and i'_n has 0. $\alpha[z]$ is a subset of every $\alpha[t'_n]$. So it is a subset of the intersection of all the $\alpha[t'_n]$, but $[z]$ is not contained in any of $\alpha[i'_n]$, so it is not contained in the union of them all. This contradicts the assumption of our theorem. Hence, z_n , for some n , must be almost equal to $(0, 0, \dots)$. This means that $[i'_n] = [t'_n]$, so $\alpha([i_1] \vee \dots \vee [i_n]) = \beta = \alpha[t_1] \cap \dots \cap \alpha[t_n]$. Now, if $[t_1] \wedge \dots \wedge [t_n]$ is strictly greater than every $[i_1], [i_2], \dots$ then the element $[t_1] \wedge \dots \wedge [t_n]$ is contained in $\beta = \alpha[t_1] \cap \dots \cap \alpha[t_n]$, but not in $\alpha[i_1] \cup \alpha[i_2] \cup \alpha[i_3] \cup \dots$ which is a contradiction to the requirement of the Theorem. \square

Corollary 11. *If $\alpha[i_1] \cap \alpha[i_2] \cap \alpha[i_3] \cap \dots = \alpha[i]$ for some $[i] \in I$ then there exists a finite number n such that $[i_1] \wedge \dots \wedge [i_n] = [i]$.*

4 Local Topological Properties of the P.I.L.

Let S_0 be a Riemann Surface. We will say that a sequence of open neighborhoods $\overline{U(1)}, \overline{U(2)}, \dots$ of z in S_0 shrinks to z if $\overline{U(i+1)} \subset U(i)$ for all i , where $\overline{U(i+1)}$ is the closure of $U(i+1)$ in S_0 , and $z = \bigcap_{i=1}^{\infty} U(i)$.

Let $sq = \{z(1), z(2), z(3), \dots\}$ be a sequence of points in S_0 . We say that a path $p : [0, 1] \rightarrow S_0$ passes through the sq in the correct way if there exists some $0 \leq t_1 \leq t_2 \leq \dots \leq 1$ such that $z(m) = p(t_m)$.

Lemma 12. *A path $p : [0, 1] \rightarrow S_0$ can pass through sq in the correct way if and only if sq converges to some point $z \in S_0$ and*

$$p\left(\lim_{m \rightarrow \infty} t_m\right) = z.$$

Proof. If sq converges to z , then we construct the path p as follows: Let $\{U(1), U(2), U(3), \dots\}$ be a sequence of open neighborhoods of z shrinking to z . Since S_0 is Hausdorff and locally compact, these sets can be chosen so that each U_i contains all the points $z(i), z(i+1), \dots$ and does not contain points $z(1), \dots, z(i-1)$. We choose any path inside $U(1)$ from $z(1)$ to $z(2)$, which we regard as a continuous map from $[0, \frac{1}{2}]$ to $U(1) \subset S_0$. Next, we choose any path inside $U(2)$ from $z(2)$ to $z(3)$, which we regard as a continuous map from $[\frac{1}{2}, \frac{3}{4}]$ to $U(1) \subset S_0$. We then proceed to choose a path inside $U(3)$ from $z(3)$ to $z(4)$ and

so on. Finally, we glue all these paths together and we map the point $1 \in [0, 1]$ to z . It is easy to check that we will obtain a continuous function from $[0, 1]$ into S_0 and that all the points of sq are contained in its image in the correct way. For the other direction of the Lemma, if there exist some $0 \leq t_1 \leq t_2 \leq \dots \leq 1$ such that $z_i = p(t_i)$ for all i , then the sequence $\{t_1, t_2, \dots\}$ converges to some $t \in [0, 1]$ and, since p is continuous, the sequence $\{z(1), z(2), z(3), \dots\}$ converges to $p(t)$. \square

Lemma 13. *For every irregular point $x \in S_\infty$, there exists some open neighborhood U of x such that for any open neighborhood V of x contained in U , there are infinitely many positive integers $n(1), n(2), \dots$ such that $V_{n(i)}$ contains some critical points of $f_{n(i)-1}$ while $(U - V)_{n(i)}$ does not contain any critical points of $f_{n(i)-1}$.*

Proof. If no such U exists, then for any open neighborhood U of x , there exists some open neighborhood $V(1) \subset U$ of x such that for infinitely many positive integers $n'(1), n'(2), \dots$, both $V(1)_{n'(i)}$ and $(U - V(1))_{n'(i)}$ contain some critical points of $f_{n'(i)-1}$. By the same logic, for the open neighborhood $V(1)$ of x there exists some open neighborhood $V(2) \subset V(1) \subset U$ of x such that for infinitely many positive integers $n''(1), n''(2), \dots$ all three sets $V(2)_{n''(i)}$ and $(V(1) - V(2))_{n''(i)}$ and $(U - V(1))_{n''(i)}$ contain some critical points of $f_{n''(i)-1}$. This process can be repeated any finite amount of times. But since every f_j has at most $(d - 1) \cdot \chi(S_0)$ [where $\chi(S_0)$ is the Euler characteristic of S_0] critical points, this process must halt after at most $(d - 1) \cdot \chi(S_0)$ repetitions. \square

The crucial result, which associates the local topology of the Plaque Inverse Limit and the dynamics of the system, is:

Theorem 14. *For every irregular point $x \in S_\infty$, there exists some open neighborhood U of x such that the open set $U - \{x\}$ has an uncountable number of path-connected components.*

Proof. Let U be an open neighborhood of x , as in Lemma 13, and let $U(1), U(2), \dots$ be open neighborhoods shrinking to x and contained in U . We can find an infinite sequence of positive integers $n(1), n(2), \dots$ such that $U(i)_{n(i)}$ contains some critical points of $f_{n(i)-1}$ while $(U - U(i))_{n(i)}$ does not contain any critical points of $f_{n(i)-1}$. For each $i = 1, 2, \dots$ let $y(i)_{n(i)} \in U(i)_{n(i)}$ be a critical point of $f_{n(i)-1}$ and for all $m < n(i)$ let $y(i)_m = f_{m+1} \circ \dots \circ f_{n(i)-1}(y(i)_{n(i)})$. Then, $y(n)_n, y(n+1)_n, y(n+2)_n, \dots$ converges to $x_n \in S_n$ for every n .

By Lemma 12, there exists a path $p_1 : [0, 1] \rightarrow S_1$, with its image totally inside U_1 , such that for some $0 \leq t_1 \leq t_2 \leq \dots \leq 1$ and for all i we get $p_1(t_i) = y(i)_1$ and $p_1(t) = x_1$ where $t = \lim_{i \rightarrow \infty} t_i$. We define the path $p_2 : [0, 1] \rightarrow S_2$ as a lift of the path p_1 to S_2 such that $p_2(t) = x_2$ and $p_2(t_i) = y(i)_2$ for all i . We define the path $p_3 : [0, 1] \rightarrow S_3$ as a lift of p_2 to S_3 such that $p_3(t) = x_3$ and $p_3(t_i) = y(i)_3$ for all i . We continue in this way until we construct the path $p_{n(1)-1}$.

Now, we define two paths $p(0)_{n(1)} : [0, 1] \rightarrow S_{n(1)}$ and $p(1)_{n(1)} : [0, 1] \rightarrow S_{n(1)}$ as any two lifts of the path $p_{n(1)-1}$ to $S_{n(1)}$ such that $p(0)_{n(1)}(t) = p(1)_{n(1)}(t) = x_{n(1)}$ and $p(0)_{n(1)}(t_i) = p(1)_{n(1)}(t_i) = y(i)_{n(1)}$ for all i while selecting two different lifts to $S_{n(1)}$ of the piece $p_{n(1)-1}[0, y(1)_{n(1)}] \rightarrow S_{n(1)-1}$. Now we lift both paths $p(0)_{n(1)}$ and $p(1)_{n(1)}$ to $S_{n(1)+1}$ while requiring that their lifts map

t to $x_{n(1)+1}$ and map t_i to $y(i)_{n(1)+1}$ for all $i > 1$. We continue this way until we construct the two paths $p(0)_{n(2)-1}$ and $p(1)_{n(2)-1}$.

We define four paths $p(0, 0)_{n(2)} : [0, 1] \rightarrow S_{n(2)}$ and $p(0, 1)_{n(2)} : [0, 1] \rightarrow S_{n(2)}$ as lifts of $p(0)_{n(2)-1} : [0, 1] \rightarrow S_{n(2)-1}$ and $p(1, 0)_{n(2)} : [0, 1] \rightarrow S_{n(2)}$ and $p(1, 1)_{n(2)} : [0, 1] \rightarrow S_{n(2)}$ as lifts of $p(1)_{n(2)-1} : [0, 1] \rightarrow S_{n(2)-1}$ to $S_{n(2)}$, while requiring that $p(*, *)_{n(2)}$ map t to $x_{n(2)}$ and map t_i to $y(i)_{n(2)}$ for all $i = 2, 3, \dots$. We now lift all four paths to $S_{n(2)+1}$ while requiring that these lifts map t to $x_{n(2)+1}$ and map t_i to $y(i)_{n(2)+1}$ for all $i > 2$. This process can be continued infinitely.

Thus, for each binary sequence sq of the above mentioned choices of lifts, we obtain a unique lift to S_∞ of the path p_1 , lying inside U , which takes t to x and which maps 0 to the point $x(sq)$ in U , with $x(sq)$ being some unique point for each binary sequence sq . Now fix any binary sequence sq and let $h : [0, 1] \rightarrow S_\infty$ be any two path inside U , which do not contain x such that $h(0) = x(sq)$. Since $[0, 1]$ is compact and S_∞ is Hausdorff and $U(1), U(2), \dots$ shrink to x , the image of h does not intersect $U(m)$ for some natural number m . Since we can always drop any finite number of $U(i)$ from our construction, we can assume that $m = 1$. Then $h(1) = x(sq')$ for some different binary sequence sq' of choices of lifts of paths if, and only if, the winding number of the loop $h_1 : [0, 1] \rightarrow (U - U(1))_1$ is nonzero. Thus, only a countable amount of different points $x(sq')$ can be connected to the point $x(sq)$ by a path in U which avoids x . So, removing x from U breaks U into an uncountable number of path-connected components. \square

We are now ready to define an important local topological invariant of the P.I.L.

Definition 15. For an open neighborhood $U \subset S_\infty$ and a critical point $c \in S_0$, we define the index of U with respect to c to be the binary sequence $ind(U, c)$ which has 1 in its n^{th} place if, and only if, $c \in U_n$.

It is clear that if $V \subset U$, then $ind(V, c) \leq ind(U, c)$.

Definition 16. For a point $x \in S_\infty$ and a critical point $c \in S_0$ of f , we define the signature of x with respect to c as:

$$sign(x, c) = \bigcap_{j=1}^{\infty} \alpha([ind(U(j), c)])$$

where $\{U(1), U(2), \dots\}$ is an arbitrary sequence of open sets in S_∞ shrinking to x .

Lemma 17. The $sign(x, c)$ does not depend on the sequence of open sets.

Proof. Let $\{U'(1), U'(2), \dots\}$ be another choice of open sets shrinking to x . Then for each $U(t)$ exists some $U'(t')$ such that $U'(t') \subset U(t)$. Thus,

$$\alpha([ind(U'(t'), c)]) \subset \alpha([ind(U(t), c)])$$

and so the signature, defined using $U'(1), U'(2), \dots$ is a subset of the signature, defined using $U(1), U(2), \dots$. But, by the same argument, the reverse is also true. Thus, these signatures are equal. \square

Lemma 18. *We have that $\text{sign}(x, c) \neq \{[0, 0, 0, \dots]\}$ if and only if x is an irregular point.*

Lemma 19. *For any $x, x' \in S_\infty$ if $\text{sign}(x, c) \cap \text{sign}(x', c)$ contains any element, other than $[0, 0, 0, \dots]$, then $x = x'$.*

Proof. If $x \neq x'$ there exists some integer number t such that $x_t \neq x'_t$. Thus, we can select disjoint neighborhoods U_t of x_t and U'_t of x'_t . So, we can construct neighborhoods U of x , and U' of x' , such that U_j and U'_j are disjoint for all $j \geq t$. Hence, $\text{ind}(U, c) \cap \text{ind}(U', c) = \{[0, 0, 0, \dots]\}$. \square

Lemma 20. *For any integer m and any point $x \in S_\infty$, we have*

$$\text{sign}(f^m(x), c) = \text{shift}_{-m}(\text{sign}(f(x), c)).$$

At this point we investigate various irregular points of P.I.L. and study their local topology.

Definition 21. *For every critical point c of f , let $P(c)$ be the closure of the orbit of c .*

The set $P(c)$ is a forward invariant, that is $f(P(c)) = P(c)$. But, in general $P(c)$ is a proper subset of $f^{-1}(P(c))$. For any point $x \in S_\infty$, the signature $\text{sign}(x, c)$ contains an element different from $[0, 0, \dots]$ only if all the coordinates x_i belong to $P(c)$ for some critical point c .

Definition 22. *A subset V of $P(c)$ is called inverse-critical with respect to c , if for any $x \in V$ and any neighborhood U of x in S_0 , we can always find a pre-image y of x under some iterate f^n of f such that y belongs to V and the connected component of $f^{-n}(U)$, which contains y , also contains c .*

It is straightforward to check that $f(V) = V$ for any inverse-critical set V . It is also straightforward to check that if V and W are inverse-critical with respect to c then $V \cup W$ is also inverse-critical with respect to c . Thus we can define:

Definition 23. *The subset $\Gamma(c)$ of $P(c)$, which is the union of all the inverse-critical sets with respect to c , is called the maximal inverse-critical set with respect to c .*

A point $x \in S_\infty$ is irregular if and only if every coordinate x_i is contained in some maximal inverse-critical set $\Gamma(c)$. In that case, the signature $\text{sign}(x, c)$ is greater than $\{[0, 0, \dots]\}$.

Theorem 24. *If some critical point c is contained in $P(c)$, then $P(c)$ is inverse-critical and for any $x_0 \in P(c)$ we can construct an irregular point $x \in S_\infty$ such that $p_0(x) = x_0$. If for some positive integer n , we have $c = f^n(c)$ then this is the case of the super-attracting cycle, which we will discuss in Theorem 26. Otherwise, $\text{sign}(x_0, c) \neq \text{shift}_k(\text{sign}(x_0, c))$ for any non-zero k .*

Proof. The case of a super-attracting cycle will be discussed in Theorem 26. Now we assume that $c \neq f^n(c)$ for all $n = 1, 2, \dots$. Thus, c is an accumulation point of $P(c)$. Since f takes a convergent sequence to a convergent sequence, this implies that every point of $P(c)$ is an accumulation point of $P(c)$.

For $t = 1, 2, \dots$, let $\{U(t)\}$ be a sequence of neighborhoods of x_0 shrinking to x_0 . We can select a pre-image x_1 of x_0 , and pre-images x_2 of x_1 , and so on,

until some pre-image x_{k_1} of x_0 so that $x_{k_1} \in P(c)$ and the lift U_{k_1} of $U(1)$ along these selected pre-images will contain c .

Again, we select a sequence of pre-images of x_{k_1} , until some pre-image x_{k_2} of x_{k_1} , such that $x_{k_2} \in P(c)$ and the lift $U(2)_{k_2}$ of $U(2)$ along these selected pre-images containing c .

This process can be continued infinitely, thus producing some $x \in S_\infty$. It is easy to see that all neighborhoods $U(t)$ of x contain an infinite amount of copies of c and that $U(t)$ shrink to x . Thus, x is irregular and $\text{sign}(x, c) \neq \{[0, 0, \dots]\}$. But any shift in indexes of x produces a different irregular point, not equal to x . Hence, $\text{sign}(x, c) \neq \text{shift}_k(\text{sign}(x, c))$ for $k \neq 0$. \square

Theorem 25. *If an irregular point x is not the invariant lift of a cycle, then for any critical point c and for any nonzero integer k we have $\text{sign}(x, c) \cap \text{shift}_k(\text{sign}(x, c)) = \{[0, 0, \dots, 0]\}$.*

Proof. If x is not the invariant lift of a cycle then given any nonzero integer k define the irregular point y by $y_{i+k} = x_i$ for all $i = 1, 2, \dots$ and $y_k = f(y_{k+1})$, $y_{k-1} = f(y_k)$, $\dots, y_0 = f(y_1)$. Then y is not equal to x . Hence, by Lemma 19,

$$\begin{aligned} & \text{sign}(x, c) \cap \text{shift}_k(\text{sign}(x, c)) \\ &= \text{sign}(x, c) \cap \text{sign}(y, c) = \{[0, 0, \dots, 0]\} \end{aligned}$$

\square

Let's review some concepts from holomorphic dynamics. See [1], [4], [8] for the concise treatment of the subject. A cycle in S_0 is a set of points $\{x_1, \dots, x_n\} \subset S_0$ such that $f(x_{i+1}) = x_i$ for all $i = 1, \dots, n$ and $f(x_1) = x_n$. If $f(x_i) \neq x_n$ for all $i = 2, \dots, n$, then n is called the period of the cycle. In that case, x_1, \dots, x_n are n distinct fixed points of f^n in S_0 . By the Chain Rule the derivative of f^n in any of these n points is equal to the product of the derivatives of f in all of them. The number $(f^n)'(x_1)$, which is independent of the choice of the local charts for the Riemann Surface, is called the multiplier of the cycle and is denoted by λ .

- If $|\lambda| < 1$ then the cycle is called attracting. In that case each x_i is inside some open set $U(i)$ such that $f(U(i+1)) = U(i)$, for all $i = 1, \dots, n-1$, and $f(U(1)) = U(n)$ and that for every point $y \in U(i)$ the sequence $y, f^n(y), f^{2n}(y), f^{3n}(y), \dots$ converges to x_i . All these sets $U(i)$ are pairwise disjoint. The collection $U(1), U(2), \dots, U(n)$ is called the immediate basin of attraction of the cycle $\{x_1, \dots, x_n\}$. It is a classical result of Fatou that any immediate basin of attraction must contain at least one critical point of f . In the special case in which one of x_i is itself a critical point and, consequently $\lambda = 0$, the cycle is called super-attracting. Any point of an attracting, but not a super-attracting cycle, possesses some open neighborhood in which the map f^n can be linearized: $f^n(z) = h^{-1}(\lambda \cdot h(z))$ for some conformal isomorphism h between this neighborhood and the unit circle. Any point of a super-attracting cycle possesses some open neighborhood in which the map f^n is conjugate to the map $z \rightarrow z^q$ for some integer $q > 1$: $f^n(z) = h^{-1}(h(z)^q)$ for some conformal isomorphism h between this neighborhood and some small circle centered at 0 of some radius $r \leq 1$.

Attracting or super-attracting cycles belong to the Fatou set of f . Topologically, a cycle of period n is attracting or super-attracting if and only if each x_i has some neighborhood $V(i)$ such that the closure in S_0 of $f^n(V(i))$ is contained in $V(i)$ and the intersection of all the $f_0^{m \cdot n}(V(i))$, where $m = 0, 1, 2, 3, \dots$, is $\{x_i\}$;

- If $|\lambda| > 1$ then the cycle is called repelling. The closure in S_0 of the set of all the repelling cycles of f is exactly equal to the Julia set of f . Topologically, a cycle of period n is repelling if and only if each x_i has some neighborhood $V(i)$ such that the closure in S_0 of the connected component $V^{-n}(i)$ of $f^{-n}(V(i))$, which contains x_i , is contained in $V(i)$, and the intersection of all the $V^{-m \cdot n}(i)$, where $m = 0, 1, 2, 3, \dots$, is $\{x_i\}$;
- If $|\lambda| = 1$ then the cycle is called neutral. There are three different sub-cases, all of which we will now review:
 - * If $\lambda = e^{2\pi i(\frac{q}{p})}$ is a p^{th} root of 1 then the cycle is called parabolic. In this case, the iterate $f^{p \cdot n}$ has a parabolic fixed point with multiplier 1. Each one of these parabolic fixed points has the Fatou-Leau flower structure. For each parabolic fixed point its multiplicity r is defined as the number of solutions of the equation $f(z) = z$, counting repetitions, at that point. In other words, $f(z) = z + a_r(z - x_0)^r + O((z - x_0)^{r+1})$ in some neighborhood of the fixed point x_0 . Parabolic fixed point of multiplicity r has $r - 1$ immediate attracting basins, each one of which must contain at least one critical point. See [2] for a concise discussion of the number of critical points in the immediate attracting basins of the parabolic cycles. Parabolic fixed points can not be linearized - there is no conformal isomorphism h between any neighborhood of the cycle and the unit circle exists such that $f^n(z) = h^{-1}(\lambda \cdot h(z))$. Parabolic cycles belong to the Julia set of f .
 - * If $\lambda = e^{2\pi i\theta}$, where $0 < \theta < 1$ is an irrational number such that in some neighborhood $V(i)$ of $x(i)$, the function f^n can be linearized: $f^n(z) = h^{-1}(\lambda \cdot h(z))$ for some conformal isomorphism h between this neighborhood and the unit circle - then each x_i is called a Siegel periodic point of f_0^n . The map f^n is a conformal automorphism of each $V(i)$, which is conjugate to a rotation of the unit circle around its center. Maximal $V(i)$, for which this conjugation holds, are called Siegel disks. The cycle $\{x_1, \dots, x_n\}$ of f is called a Siegel cycle. Siegel disks of f^n belong to the Fatou set of f . Topologically, a cycle of period n is Siegel if, and only if, each x_i has some neighborhood $V(i)$, such that f^n is an automorphism of $V(i)$.
 - * If $\lambda = e^{2\pi i\theta}$, where $0 < \theta < 1$ is an irrational number such that, in any neighborhood $V(i)$ of x_i , the function f^n can not be linearized, then each x_i is called a Cremer periodic point of f^n . The cycle $\{x_1, \dots, x_n\}$ of f is called a Cremer cycle. The points of the Cremer cycle are always accumulation points of all the iterations of f of some [at least one] critical point of f .

Let $(x_1, x_2, \dots) \in S_\infty$ be the invariant lift of the cycle $\{x_1, \dots, x_n\}$ of the dynamical system $f : S_0 \rightarrow S_0$. In other words, $x_{i+m \cdot n} = x_i$ for all $1 \leq i \leq n$ and $m = 1, 2, \dots$

The third point of the following theorem was also proven on [3], here we present another proof.

Theorem 26. .

- If x is the invariant lift of a repelling cycle then x is a regular point and

$$\text{sign}(x, c) = \{[0, 0, \dots]\}$$

for every critical point c ;

- If x is the invariant lift of a Siegel cycle, then x is a regular point and $\text{sign}(x, c) = \{[0, 0, \dots]\}$ for every critical point c ;
- If x is the invariant lift of either an attracting, a super-attracting cycle, a parabolic cycle or a Cremer cycle, then x is an irregular point and for some critical points c_1, \dots, c_m , the signature

$$\text{sign}(x, c_j) = \text{shift}_{k(j)}(\alpha[sq(n)])$$

where $0 \leq k(j) < n$ are some integers and the binary sequence $sq(n)$ has 1 in places $n, 2n, 3n, \dots$ and 0 everywhere else. For all other critical points, the signature at x is $\{[0, 0, \dots]\}$.

Proof. If x represents a repelling cycle, or a Siegel cycle, then x_1 has some neighborhood $U_1 \subset S_1$ such that the pre-images U_2, \dots, U_n of U_1 along x_2, \dots, x_n do not contain any critical points and such that each U_{i+n} , for all i , is a subset of U_i . Thus, none of U_i will contain any critical points and so $\text{sign}(x, c) = \{[0, 0, \dots]\}$ for every critical point c .

If x represents an attracting or a super-attracting cycle then (see [8], [4], [1]) for any neighborhood U_1 of x_1 exists some positive integer q such that for some critical point c_1 and some integer $1 \leq k_1 \leq n$, the pre-images U_{t-n+k_1} of U_1 along x will contain some critical point c_1 for all $t \geq q$. The pre-images U_i and U_j of U_1 along x are disjoint for $(i \neq j) \bmod n$. Hence, we get that $\text{sign}(x, c_1) = \text{shift}_{k_1}(\alpha[sq(n)])$. But if $k_1 = n$ we can make $k_1 = 0$ and substitute $\text{shift}_n(\alpha[sq(n)])$ by $\text{shift}_0(\alpha[sq(n)])$.

On the other hand, if some critical point c_2 is contained in the immediate basin of attraction of the cycle, then for any neighborhood U_1 of x_1 there exists some integer number q such that all the pre-images U_{q+t-n} of U_1 along x contain c_2 for all $t = 0, 1, 2, \dots$. Now, let $k_2 = q \bmod n$. We get that $\text{sign}(x, c_2) = \text{shift}_{k_2}(\alpha[sq(n)])$.

If x is the invariant lift of a parabolic cycle then (see [8], [4], [1], [2]), the immediate attracting basins of some of x_i , where $1 \leq i < n$, contains some critical points c_i . Hence, the first argument brought above for the attracting cycles, applies here too and for some critical points c_1, \dots, c_m the signature $\text{sign}(x, c_j) = \text{shift}_{k_j}(\alpha[sq(n)])$. On the other hand, suppose that for some critical point c , the signature $\text{sign}(x, c)$ contains some non-zero element $[a] \in I$. Let the binary sequence a be any representative of the class $[a]$. Let $U(t)$, where $t = 1, 2, \dots$, be a sequence of neighborhoods of x which shrinks to x . Then for each $U(t)$ and for almost every $i = 1, 2, \dots$, such that the i^{th} entry of a is 1, the i^{th} projection $U(t)_i$ of $U(t)$ contains the critical point c .

If we take a sequence of neighborhoods $V(t)$ of x , which shrinks to x , where each $V(t)_i$ is a copy in S_i of $U(t)_{i+n}$, then, again, for almost every $i = 1, 2, \dots$,

such that the i^{th} entry of a is 1, the i^{th} projection $V(t)_i$ of $V(t)$ contains the critical point c . Hence, $[a] = shift_n([a])$. So, up to a finite amount of entries, $a = shift_n(a)$. So the entire binary sequence a , except in a finite amount of places, is just an infinite periodic copy of some binary sequence of length n . But for all t greater than some positive integer the copies of $U(t)_i, U(t)_{i+1}, U(t)_{i+2}, \dots, U(t)_{i+n-1}$ in S_0 are all pairwise disjoint and so, at most one of them can contain c . Thus, $sign(x, c) = shift_k(\alpha[sq(n)])$ for some $0 \leq k < n$.

If x is the invariant lift of a Cremer cycle then let Υ be any neighborhood of x in S_∞ . We need to show that for some c and for an infinite amount of different positive integers i , the point c belongs to the i^{th} projection Υ_i of Υ . If this is not the case, we can select some small enough Υ such that none of Υ_i contains any critical point. Let $U(i)$ be the copy of Υ_i in S_0 . All the sets $U(i), U(i+n), U(i+2n), U(i+3n), \dots$ contain the common point $x(i)$ - the copy of x_i in S_0 . Hence, the union $U \subset S_0$ of all the $U(i), U(i+n), U(i+2n), U(i+3n), \dots$ is path-connected. Let $V = f^n(U)$. Then V is also path-connected. We have that $U \subset V$. By our construction, U does not contain any critical points of f and, consequently, of f^n . Hence, $f^n : U \rightarrow V$ is a covering map with a degree t between 1 and d^n . If U is just the whole Riemann Sphere with two points removed, then $V = U$ and f^n is conjugate to $z^{\pm t}$ and, thus, has no Cremer points. So, U and V are conformally hyperbolic. If $U = V$ there exists a conformal covering map ψ from the unit disk on U , which takes 0 to $x(i)$. Then the map $\psi^{-1} \circ f^n \circ \psi$ from the unit disk onto itself, in which ψ^{-1} is constructed by requiring that it takes $x(i)$ to 0, is a well-defined conformal automorphism of the Unit Circle, which has its derivative equal to λ at 0. Hence, it must be a rotation - the multiplication by λ . Consequently, in some neighborhood of $x(i)$ in S_0 the function f^n is conjugate to a rotation and x_i is not a Cremer point of f^n . Therefore, U is strictly smaller than V . As a corollary of the Schwarz-Pick-Ahlfors theorem (see pages 22-24 in [8]), the inclusion map of U into V strictly decreases the hyperbolic distance $dist_U(x, y) > dist_V(x, y)$ for all the "close-enough" points x and y , $x \neq y$, in U . On the other hand, by the Schwarz-Pick-Ahlfors theorem, the function $f^n : U \rightarrow V$ is a local isometry and for all the "close-enough" points x and y in U we have $dist_V(f^n(x), f^n(y)) = dist_U(x, y)$. Thus, for all the "close-enough" points x and y , $x \neq y$, in U we get $dist_V(f^n(x), f^n(y)) > dist_V(x, y)$. So f^n is repelling near x_i , which again contradicts the fact the x_i is a Cremer point of f^n . Thus, for some critical point c and for any neighborhood U of x , c is contained in an infinitely many projections U_i of U . So, due to Corollary 11, the signature $sign(x, c)$ is greater than $\{[0, 0, \dots]\}$. Now we repeat the same argument as for the parabolic cycle to prove that all the nontrivial signatures in this case are $sign(x, c) = shift_k(\alpha[sq(n)])$. \square

See [5] for a discussion of cases in which a boundary component of a Siegel disk or of a Herman ring must contain a critical point and when that boundary component must be a Jordan curve.

Theorem 27. *If a boundary component of a Siegel Disk or of a Herman ring is a Jordan curve and contains a critical point c , then this critical point is the accumulation point of its forward orbit, but is not equal to any $f^n(c)$ for any $n > 0$. Thus, our Theorem 24 applies in this case. In that case $P(c)$ equals to that entire boundary component.*

Proof. On any boundary component of a Siegel disk or of a Herman ring, which is a Jordan curve, the map f , by the Caratheodory extension theorem, is conjugated by a continuous homeomorphism to a rotation of the circumference of a unit circle by an angle which is an irrational multiple of 2π . Thus, the forward orbit of any critical point c , belonging to this boundary component, is dense in this boundary component. \square

References

- [1] Alan F. Beardon. Iteration of Rational Functions. Springer-Verlag, 1991.
- [2] Walter Bergweiler. On the number of critical points in parabolic basins, Erg. Th. and Dyn. Sys. (2002), 22, 655 to 669.
- [3] Carlos Cabrera and Tomoki Kawahira. On the natural extension of a map with a Siegel or Cremer point. arXiv 1103.2905 [math.DS], 15 Mar 2011.
- [4] Lennart Carleson and Theodore W. Gamelin. Complex Dynamics, Universitext / Universitext: Tracts in Mathematics (July 1993).
- [5] Arnaud Cheritat and Pascale Roesch. Herman's Condition and Siegel Disks of Polynomials, arXiv 1111.4629 [math.DS], 20 Nov 2011.
- [6] Mikhail Lyubich and Yair Minsky. Laminations in holomorphic dynamics, J. Diff. Geom. 47 (1997), 17 to 94.
- [7] Mikhail Lyubich, Michael Yampolsky. Holomorphic dynamics and renormalization: a volume in honour of John Milnor, American Mathematican Soc., (September 2008).
- [8] John W. Milnor. Dynamics in One Complex Variable, Princeton University Press (2006).
- [9] Saeed Zakeri. On critical points of proper holomorphic maps on the unit disk, Bulletin of the London Mathematical Society (1998), 30, 62 to 66.